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New results on edge-bandwidth

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Abstract

The edge-bandwidth problem is an analog of the classical bandwidth problem, in which one has to label the edges of a graph by distinct integers such that the maximum difference of labels of any two incident edges is minimized. We prove tight bounds on the edge-bandwidth of hypercube and butterfly graphs, and complete k-ary trees which extends and improves on previous known results.

Keywords: Bandwidth; Butterfly; Edge-bandwidth; Hypercube

1. Introduction

Let $G=(V,E)$ be a simple graph with $|V|=n$ and $|E|=m$. Let f be a bijection from V to the set $\{1,2,3,\ldots,n\}$, called a labelling of vertices of G. The *bandwidth* of G is defined to be

$$B(G) = \min_{f} \max_{(u,v)\in E} |f(u) - f(v)|,$$

where the minimum is taken over all possible labellings f of G. There are several motivations for studying the bandwidth problem: sparse matrix computations, representing data structures by linear arrays, VLSI layouts and mutual simulations of interconnection networks, see surveys [3,4,17]. The problem is NP-hard and is inapproximable by any multiplicative constant even for the class of caterpillar graphs [19], unless $P=NP$.

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Bandwidths are known only for a few infinite families of graphs including hypercubes [9], complete trees [10] and various mesh-like graphs, see [3,10,11,14,16]. Lower bound techniques are surveyed in [20].

The edge-bandwidth problem goes back to the work of Hwang and Lagarias [12]. It is defined as an analog of the bandwidth problem where instead of vertices one labels the edges. More formally, let g be a bijection from E to the set $\{1,2,3,\ldots,m\}$, called a labelling of edges of G. The *edge-bandwidth* of G is

$$B'(G) = \min_g \max\{|g(a) - g(b)| : a,b \in E, a,b \text{ are incident}\},$$

where the minimum is taken over all possible labellings g of edges of G. Grünwald and Weber [7,8] determined edge-bandwidths for complete binary trees, complete and complete bipartite graphs. Bezrukov et al. [2] considered the edge-bandwidth of the n-dimensional hypercube graph $\mathcal{Q}_n$ and showed estimations:

$$2^{n-1} + 2^{n-2} \leqslant B'(\mathcal{Q}_n) \leqslant 2\left\lceil \frac{n}{2} \right\rceil \binom{n}{\lfloor \frac{n}{2} \rfloor} - 1. \tag{1}$$

Recently, Tao Jiang et al. [13] rediscovered the edge-bandwidth for $K_n, K_{n,n}$ and found an exact result for caterpillars. In a subsequent paper Eichhorn et al. [5] computed edge-bandwidths of all theta graphs. Let $L(G)$ denote the line graph of G i.e. the graph whose vertices are edges of G and two vertices are adjacent if and only if the edges were incident in G. Then by the above definition

$$B'(G) = B(L(G)). \tag{2}$$

The aim of this paper is to prove several new results on the edge-bandwidth for typical graphs. Section 2 contains useful upper and lower bounds on bandwidths. In Section 3 we essentially improve the lower bound in (1). Tight bounds on the edge-bandwidth for butterfly graphs and complete k-ary trees are in Sections 4 and 5, respectively. The technique used to achieve the upper bound on the edge-bandwidth for butterflies is used also to improve the previously known results on the bandwidth of butterflies. In the last section, we discuss a possible further research.

2. General bounds

First we mention two powerful lower bound methods for estimating the bandwidth and then prove a new relation between the edge-bandwidth and bandwidth.

Let $G=(V,E)$ be a graph. For $S \subseteq V$, let

$$\partial(S) = \{v \in V - S|\ (u,v) \in E, u \in S\}.$$

Harper [9] in his seminal work on the bandwidth of the hypercube graph implicitly proved:

Theorem 1. *For any* k, $0 \leqslant k \leqslant |V|/2$

$$B(G) \geqslant \min_{\substack{S \\ |S|=k}} \max\{|\partial(S)|, |\partial(V-S)|\}.$$

Another useful estimation is, see e.g. [20]:

Lemma 2. *Let* H *be a graph on* p *vertices of diameter* $\operatorname{diam}(H) > 0$. *Then*

$$B(H) \geqslant \left\lceil \frac{p-1}{\operatorname{diam}(H)} \right\rceil. \tag{3}$$

Jiang et al. [13] proved that

$$B'(G) \leqslant 2tB(G) + t - 1,$$

where t denotes the arboricity of G. Let Δ denote the maximum degree of G. As $t \leqslant \Delta$ we immediately have

$$B'(G) \leqslant 2\Delta B(G) + \Delta - 1,$$

We prove the following:

Theorem 3. *For the edge-bandwidth of any graph*

$$B'(G) \leqslant \Delta B(G) + \Delta - 1.$$

Proof. Consider an optimal labelling of G with respect to the bandwidth measure. Identify the vertices with their labels. Let d_i be the degree of the vertex i.

Label edges incident to 1 by $1, 2, \ldots, d_1$.

Label unlabelled edges incident to 2 by $d_1 + 1, d_1 + 2, \ldots, d_1 + x_2$, where $x_2 \leqslant d_2$.

Label unlabelled edges incident to $i \leqslant n-1$ by $d_1 + x_2 + x_3 + \cdots + x_{i-1} + 1, \ldots, d_1 + x_2 + x_3 + \cdots + x_i$, where $x_i \leqslant d_i$.

Now we check the labelling. Let (i, j) and (j, k) be any pair of incident edges of G. Assume that $i < j < k$. Clearly the label of (i, j) is at least $d_1 + x_2 + x_3 + \cdots + x_{i-1} + 1$ and the label of jk is at most $d_1 + x_2 + x_3 + \cdots + x_{i-1} + x_i + \cdots + x_j$. Hence the difference of labels of edges (i, j) and (j, k) is at most

$$x_i + \cdots + x_j - 1 \leqslant (j - i + 1)\Delta - 1 \leqslant (B(G) + 1)\Delta - 1.$$

Assume now $i, k < j$. The proof is similar. □

3. The hypercube graph

In this section, we essentially improve the lower bound from [2] for the edge-bandwidth of the hypercube graph. In the n-dimensional hypercube $\mathcal{Q}_n$, the vertices are all binary strings of length n, and two vertices are adjacent if and only if they differ in exactly one position.

Theorem 4. *The edge-bandwidth of the n-dimensional hypercube satisfies*

$$B'(\mathcal{Q}_n) \geqslant \frac{n}{4}\binom{n}{\lceil \frac{n}{2} \rceil}.$$

Proof. Consider the graph $L(\mathcal{Q}_n)$. It has $n2^{n-1}$ vertices. We prove that for any $n2^{n-2}$-vertex set $S \subset V$

$$\max\{|\partial(S)|, |\partial(V-S)|\} \geqslant \frac{n}{4}\binom{n}{\lceil \frac{n}{2} \rceil},$$

which—in combination with Theorem 1—will imply the lower bound.

Color the vertices of S by red and the vertices of $V-S$ by white. Note that $L(\mathcal{Q}_n)$ is a union of 2^n n-cliques, where the edge set of $L(\mathcal{Q}_n)$ is a disjoint union of the edges of the cliques. Let R, W and M be the set of all red, white and mixed cliques, respectively. Clearly, $|R|+|W|+|M|=2^n$. For a mixed clique $c \in M$, let x_c denote the number of its red vertices, $1 \leqslant x_c \leqslant n-1$.

Since each node is shared by exactly two n-cliques, it is straightforward to observe that

$$|\partial(S)| \geqslant \frac{1}{2}\sum_{c\in M}(n-x_c) \tag{4}$$

and similarly

$$|\partial(V-S)| \geqslant \frac{1}{2}\sum_{c\in M} x_c. \tag{5}$$

Then

$$\max\{|\partial(S)|, |\partial(V-S)|\} \geqslant \tfrac{1}{2}(|\partial(S)| + |\partial(V-S)|) \geqslant \tfrac{1}{4}|M|n.$$

Distinguish two cases:

(i) If $|M| \geqslant \binom{n}{\lceil \frac{n}{2} \rceil}$ then we are done.

(ii) Assume $|M| < \binom{n}{\lceil \frac{n}{2} \rceil}$. We show that this case is impossible.

Summing up the numbers of red vertices in red cliques and mixed cliques one has

$$|R|n + \sum_{c\in M} x_c = n2^{n-1} \tag{6}$$

as every red vertex was counted twice. Similarly for the number of white vertices

$$|W|n + \sum_{c\in M}(n-x_c) = n2^{n-1}.$$

We can assume that

$$\sum_{c\in M} x_c \leqslant \frac{|M|n}{2}, \tag{7}$$

otherwise we change the role of the red and white vertices. By combining (7) and (6) we get

$$2^{n-1} - \frac{1}{2}\binom{n}{\lceil \frac{n}{2} \rceil} < |R| < 2^{n-1}. \tag{8}$$

Now consider again the original hypercube $\mathcal{Q}_n$. Clearly, there is a one-to-one correspondence between n-cliques in $L(\mathcal{Q}_n)$ and vertices in $\mathcal{Q}_n$, according to the line graph operation. In other words, a vertex v in $\mathcal{Q}_n$ corresponds to a clique in $L(\mathcal{Q}_n)$, created on the edges of $\mathcal{Q}_n$ incident to v. Let R' be the set of all vertices in $\mathcal{Q}_n$ which correspond to cliques in R. Define similarly the sets W' and M'. Then $|R'| = |R|, |W'| = |W|$ and $|M'| = |M|$. Observe that $\partial(R') \subseteq M'$ and hence

$$|\partial(R')| \leqslant |M'| = |M| < \binom{n}{\lceil \frac{n}{2} \rceil}. \tag{9}$$

Frankl [6] proved the following useful estimation: Let A be a subset of the vertices of $\mathcal{Q}_n$. If

$$|A| = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{r+1} + \binom{y}{r}$$

for an integer r and a real y, then

$$|\partial(A)| \geqslant \binom{n}{r} + \binom{y}{r-1} - \binom{y}{r}.$$

Because (8) implies

$$2^{n-1} - \frac{1}{2}\binom{n}{\lceil \frac{n}{2} \rceil} < |R'| < 2^{n-1}$$

we have

$$|R'| = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{\lceil \frac{n}{2} \rceil + 1} + \binom{y}{\lceil \frac{n}{2} \rceil}$$

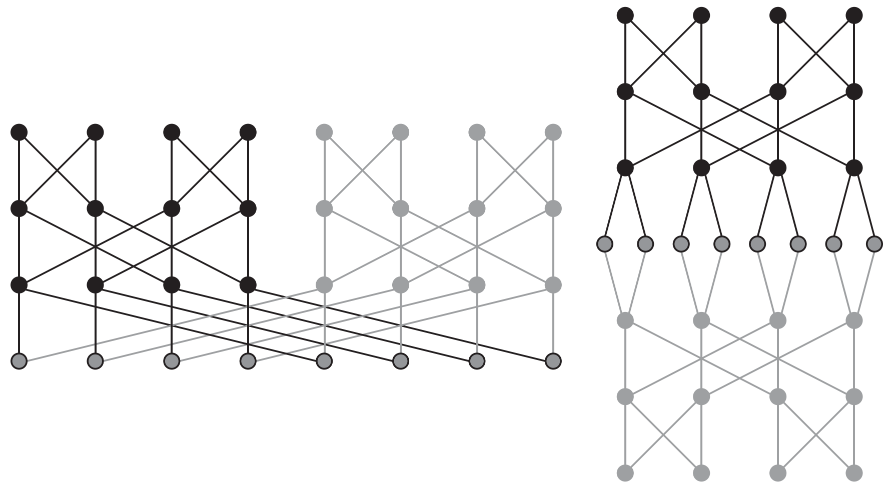
for some real $y, \lceil n/2 \rceil \leqslant y < n$. By applying the Frankl's result we get

$$|\partial(R')| \geqslant \binom{n}{\lceil \frac{n}{2} \rceil} + \binom{n}{\lceil \frac{n}{2} \rceil - 1} - \binom{y}{\lceil \frac{n}{2} \rceil} \geqslant \binom{n}{\lceil \frac{n}{2} \rceil},$$

which contradicts to (9). □

4. The butterfly graph

In this section, we present upper and lower bounds on the edge-bandwidth for butterflies. Using the same technique, we improve also the previously known result on the upper bound for the bandwidth of butterflies. The n-dimensional butterfly graph $\mathcal{B}_n$ has vertices $[i, w]$, where w is a binary string of length n and i is an integer in the range



(a) (b)

Fig. 1. Two different representations of $\mathscr{B}_3$.

from 0 to n. The vertex $[i,w]$ is adjacent to $[i+1,w']$ if and only if either $w=w'$ or $w=\beta_1,\beta_2,\dots,\beta_{i-1},\beta_i,\beta_{i+1},\dots,\beta_{n-1}$ and $w'=\beta_1,\beta_2,\dots,\beta_{i-1},\overline{\beta_i},\beta_{i+1},\dots,\beta_{n-1}$. The n-dimensional butterfly has $2^n(n+1)$ vertices and $2^{n+1}n$ edges. Its diameter is $2n$. The butterfly graph represents the standard interconnection network of parallel computers [15], especially suitable for sorting and the Fast Fourier Transform, and it is usually graphically represented as in Fig. 1a. However, it is possible to highlight the symmetry of butterfly graphs with respect to its last level, as depicted in Fig. 1b. In the following, we will use just this representation.

Before proving the bound on the edge-bandwidth of the butterfly, we need to do some preliminary observations.

First note that the n-dimensional butterfly network $\mathscr{B}_n$ can be covered by 2^{n+1} edge-disjoint complete binary trees as follows:

- two trees T_{n+1} having $n+1$ edge levels, sharing their leaves;
- for any $i=3,\dots,n$, 2^{n+1-i} trees T_i having i levels, sharing their leaves with internal vertices of some tree T_j, $j>i$, (and their internal vertices with some T_k, $k<i$).

Since we use the butterfly representation of Fig. 1b, in the following we consider only half of all these trees, since the other half is symmetrical (see Fig. 3a).

In view of the previous decomposition in trees, we need to describe the line graph of a complete binary tree to construct $L(\mathscr{B}_n)$. Let T_n be the complete binary tree of depth n. The graph $L(T_n)$ is constructed from two binary trees of type T_{n-1} in the following way: in every non-leaf vertex of each T_{n-1} join its children and finally join the roots of both trees. See Fig. 2 where T_4 and $L(T_4)$ are depicted.

The graph $L(T_n)$ has $2^{n+1}-2$ vertices and diameter $2n-1$. The vertices of $L(T_n)$ are divided into levels $1,2,3,\dots,n$, starting from the top.

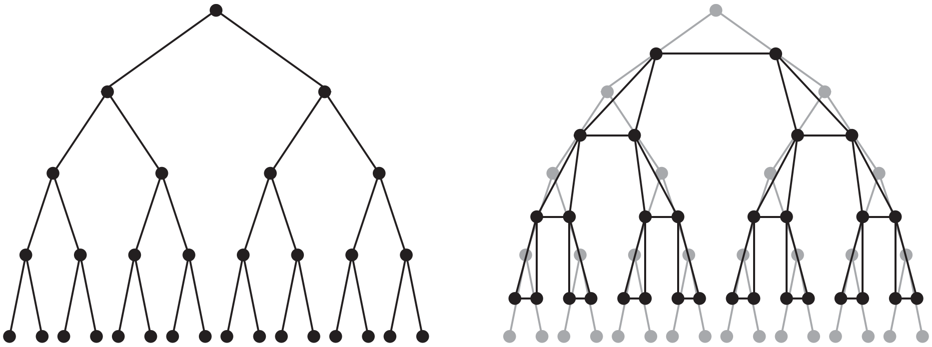


Fig. 2. T_4 and $L(T_4)$.

Observe that $L(T_n)$ consists of two equal subgraphs, G_L and G_R, connected by one horizontal edge. Although G_L (G_R) is not a tree, in the following we will use anyway the notation of trees; e.g. we call "leaves" the vertices on the last level, "parent" of a vertex v the vertex connected to v and lying on the previous level, and so on.

Now we are ready to prove the following theorem:

Theorem 5. *The edge-bandwidth of the n-dimensional butterfly satisfies*

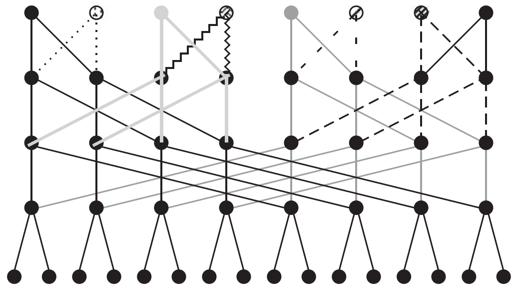
$$2^n \leqslant B'(\mathscr{B}_n) \leqslant \tfrac{5}{4}\, 2^n.$$

Proof. *Lower bound.* The lower bound follows from (3) by noting that $L(\mathscr{B}_n)$ has $2^{n+1}n$ vertices and the diameter of $2n$.

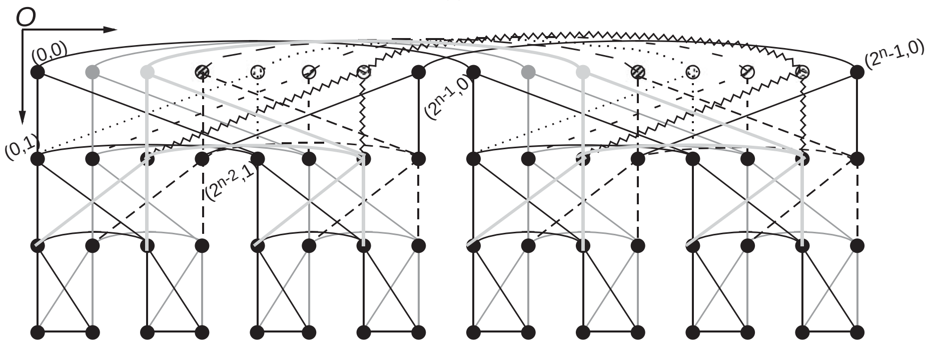
Upper bound: We prove the upper bound by giving a feasible labelling for the line graph of the n-dimensional butterfly network $L(\mathscr{B}_n)$. We construct a graphical representation of $L(\mathscr{B}_n)$ (in fact, of half of it) with vertices with integer coordinates exploiting the line graph of the complete binary tree. This representation induces an ordering on the vertices of each level and we can follow this order to label the vertices of $L(\mathscr{B}_n)$. The representation of $L(\mathscr{B}_n)$ in the plane is constructed in the following way (for an intuition, see Fig. 3b), putting the origin of the axes on the top left corner with the x-axis being directed to the right and the y-axis being directed down:

- put vertices of the line graph of the biggest tree in $\mathscr{B}_n$in the plane in the following way:
  - ∘ put the root of G_L at coordinates $(0,0)$;
  - ∘ put the root of G_R at coordinates $(2^{n-1},0)$;
  - ∘ given an already placed vertex with coordinates $(x,i-1)$, $i=1,\dots,n-1$, put its left child at coordinates (x,i);
  - ∘ given an already placed vertex with coordinates $(x,i-1)$, $i=1,\dots,n-1$, put its right child at coordinates $(x+2^{n-1-i},i)$.
- While all the trees in $\mathscr{B}_n$ have not been considered:

  Consider the biggest tree T in $\mathscr{B}_n$ not considered yet; let h be the height of T; observe that the leaves of the line graph of T have already been put in the plane, since they correspond to vertices shared with higher trees;



(a)



(b)

Fig. 3. (a) Half of $\mathscr{B}_4$, where the tree covering is highlighted. (b) A graphical representation of half of $L(\mathscr{B}_4)$.

put vertices of the line graph of T in the plane in the following way:

- for each vertex of $L(T)$ at level $h-1$:
  let v_1 and v_2 be the children of v in $L(T)$ already drawn;
  if v_1 and v_2 have coordinates (x_1, h) and (x_2, h), respectively, $x_1 < x_2$, then let $(x_2, h-1)$ be the coordinates of v;
- for all levels j from $h-2$ downto 1:
  for all vertices v of $L(T)$ at level j: let v_1 and v_2 be the children of v in $L(T)$ at level $j+1$; if v_1 and v_2 have coordinates $(x_1, j+1)$ and $(x_2, j+1)$, respectively, $x_1 < x_2$, then let (x_1, j) be the coordinates of v.

Once all vertices in $L(\mathscr{B}_n)$ have been laid out, we can easily add all its edges (see Fig. 3b) and do the following observations:

Observation 1. *Edges connecting vertices on the same level j are incident to vertices whose x-coordinates differ by 2^{n-1-j}.*

Observation 2. *The graphical representation of the set of edges connecting vertices at level j with vertices at level $j+1$ is as the usual butterfly-like set of edges between levels $n-1-j$ and $n-2-j$ and its cross width is 2^{n-j-2}.*

Now we consecutively label in increasing fashion all vertices of $L(\mathscr{B}_n)$ from left to right, from level 0 to level $n-1$, and we prove that the bandwidth of this labelling is $\frac{5}{4}2^n$.

Observe that each level contains exactly 2^n vertices of $L(\mathscr{B}_n)$. Consider now the general edge $e=(v,w)$ in $L(\mathscr{B}_n)$. Then: either v and w are on the same level, or v is on level j and w is on level $j+1$ (the inverse is symmetrical). If v and w are on the same level, then their labels differ by at most 2^{n-1} as it follows from Observation 1, applied with $j=0$. On the other hand, if v and w lie on consecutive levels—exploiting Observation 2—then it is easy to see that the biggest difference between their labels happens when the x-coordinate of v is smaller than the x-coordinate of w and $j=0$; in such a case the labels differ by the size of a whole level plus the maximum cross width, i.e. by $2^n+2^{n-2}=\frac{5}{4}2^n$. □

Concerning the vertex bandwidth of $\mathscr{B}_n$, Barth et al. [1] proved the following bounds for the bandwidth of the n-dimensional butterfly graph:

$$2^{n-1} \leqslant B(\mathscr{B}_n) \leqslant 3 \cdot 2^{n-1}.$$

By exploiting the same technique of row by row labelling, we improve the upper bound:

Theorem 6. *The bandwidth of the n-dimensional butterfly satisfies*

$$B(\mathscr{B}_n) \leqslant 2^n.$$

Proof. Consider the drawing of $\mathscr{B}_n$ from the Fig. 1b. Label the vertices by $1,2,3,\ldots,2^n(n+1)$ in the row by row manner starting from the left top vertex. One can easily see that the maximum difference is 2^n. □

We conjecture equality in the above bounds.

5. Complete k-ary trees

In this section, we give an asymptotically optimal estimation for the edge-bandwidth of the complete k-ary tree, $k \geqslant 3$. Let $\mathscr{T}_{k,n}$ denote the complete k-ary tree of the depth n. Define $G_{k,n}=L(\mathscr{T}_{k,n})$. The graph $G_{k,n}$ has $(k^{n+1}-k)/(k-1)$ vertices and diameter $2n-1$.

Theorem 7. *The edge-bandwidth of the complete k-tree of depth n satisfies*

$$\frac{k^{n+1}-2k+1}{(2n-1)(k-1)} \leqslant B'(\mathscr{T}_{k,n}) \leqslant k\left\lceil \frac{k(k^{n-1}-1)}{(2n-2)(k-1)} \right\rceil + k - 1. \quad (10)$$

Proof. *Lower bound*: A lower bound is given immediately by inequality (3)

$$B'(\mathscr{T}_{k,n}) \geqslant \frac{k^{n+1}-2k+1}{(2n-1)(k-1)}. \quad (11)$$

Upper bound: Note that every k incident edges on the same edge level in $\mathscr{T}_{k,n}$ induce a k-clique in $G_{k,n}$. By shrinking every such clique into a single vertex and by removing multiple edges we get $\mathscr{T}_{k,n-1}$. According to [18]

$$B(\mathscr{T}_{k,n-1}) = \left\lceil \frac{k(k^{n-1}-1)}{(2n-2)(k-1)} \right\rceil.$$

Now multiply every label by k and expand $\mathscr{T}_{k,n-1}$ back to $G_{k,n}$. Label the vertices of a clique by $lk, lk-1, lk-2, \dots, lk-k+1$ if the corresponding vertex in $\mathscr{T}_{k,n-1}$ was labelled by l. Finally, take any two adjacent vertices in $G_{k,n}$. Let the vertices belong to cliques which correspond to vertices labelled by l and l' in $T_{k,n-1}$, where $l \geqslant l'$. Then

$$\begin{aligned} B(G_{k,n}) = B'(\mathscr{T}_{k,n}) &\leqslant lk - (l'k - k + 1) \leqslant k(l - l') + k - 1 \\ &\leqslant k \left\lceil \frac{k(k^{n-1}-1)}{(2n-2)(k-1)} \right\rceil + k - 1. \end{aligned}$$

By combining the bounds in (11) and the last inequality we get the result. □

6. Conclusion

We essentially improved a lower bound for the edge-bandwidth of the hypercube graph and gave tight estimations for the butterfly graph and the complete k-ary trees, $k \geqslant 3$. Determining the exact values remains an open problem. Another interesting open question is the edge-bandwidth of the $m \times n$ grid. We conjecture that the optimal value is $2n-1$, for $m \geqslant n$.

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